Assessing pupil knowledge of the sequential structure of numbers

David Ellemor-Collins & Robert Wright

Abstract

Research on children's mental strategies for multidigit addition and subtraction identifies two categories of strategy. Collections-based strategies involve partitioning numbers into tens and ones, and can be modeled with base-ten materials. Sequence-based strategies involve keeping one number whole, and using the sequential structure of numbers. They can be modelled as jumps on an empty number line. Studies have found sequence-based strategies to be more successful, and to correlate with more robust arithmetic knowledge, particularly among low-attaining pupils. Studies also suggest that sequence-based strategies and sequential structure are not explicitly developed in many primary mathematics classrooms. This report draws on results from a three-year project which has the goal of developing pedagogical tools for intervention in the number learning of low-attaining third- and fourth-graders (8- to 10-year-olds). These tools include assessment tasks to inform intervention. The report focuses on four groups of assessment tasks that collectively enable detailed documenting of pupils' knowledge of the sequential structure of numbers. Tasks and pupils' responses are described in detail. Some examples follow. When asked to count back from 52, pupils said, '52, 51, 40, 49, 48, and so on'. When asked to count back by tens from 336, pupils had difficulty continuing after 326. Thus teen numbers in the hundreds (316) presented particular difficulties. Pupils had difficulty saying the number that is ten less than 306. Pupils had difficulty with locating the numbers 50, 25, 62 and 98 on a number line on which zero and 100 were marked. The report provides insight into assessing knowledge of sequential structure and argues that this is important basic number knowledge.

MULTI-DIGIT ADDITION and subtraction is an important aspect of number sense and mental computation (Anghileri, 2000; Thompson & Smith, 1999). As well, this topic is important because it provides a basis for more advanced arithmetic. This article draws from a current three-year project focusing on the development of pedagogical tools to support intervention in the number learning of low-attaining third- and fourth-graders (8- to 10-year-olds). These tools include schedules of diagnostic assessment tasks and instructional procedures. This paper focuses on some of the assessment tasks which enable assessment of knowledge of the sequential structure of numbers. Developing significant knowledge of the sequential structure of numbers provides an important basis for multi-digit addition and subtraction (Beishuizen & Anghileri, 1998). The paper will:

1. elaborate the term 'sequential structure of numbers';
2. review literature relevant to the sequential structure of numbers;
3. set out relevant diagnostic assessment tasks; and
4. describe the range of low-attaining pupils' responses to those tasks.

Sequential structure of numbers

This paper discusses low-attaining pupils' knowledge and use of what we call the 'sequential structure of numbers'. By 'sequential structure of numbers' we refer to the decade-based structures in the linear sequence 1, 2, 3, … 99, 100, 101…. Specifically, this number sequence consists of a sequence of decades, which can be further organised in a sequence of hundreds. The decade numbers (10, 20, 30,…) are reference points in the sequence, at even intervals of ten. Each decade follows the same pattern as, for example, '20, 21, 22,... 28, 29, 30'. By the neat symmetry in this sequence, a pair of numbers such as 18 and 28, or 71 and 81, is always ten steps apart. Referring to the sequential structure of numbers, 57 can be regarded as one after 56, seven after 50, three before 60, or 10 after 47.

We can also describe 'collections-based' structures in multidigit numbers. These involve thinking of numbers in terms of collections of ones, tens, hundreds and so on. For example, 57 can be constructed as fifty and seven, or as 7 ones and 5 tens.

Literature review

Emphasis on mental computation

In the last 15 years, research and curriculum reforms in a range of countries highlight a renewed emphasis on mental computation with multidigit numbers (Beishuizen & Anghileri, 1998; Thompson, 1997). An early emphasis on mental strategies, rather than formal written algorithms, may better support number sense and conceptual understanding of multidigit numbers, and support development of important connections to related knowledge (Askew, Brown, Rhodes, Wiliam & Johnson, 1997; Hiebert & Wearne, 1996; McIntosh, Reys & Reys, 1992; Sowder, 1992; Yackel, 2001). Mental computation can also stimulate the development of numerical reasoning and flexible, efficient computation (Anghileri, 2001; Treffers, 1991).

Mental strategies: 'sequence-based' and 'collections-based'

In response to the emphasis on mental computation, research projects in several countries focused on pupils' informal mental strategies for multi-digit addition and subtraction (Beishuizen, Van Putten & Van Mulken, 1997; Cobb *et al.*, 1997; Cooper, Heirdsfield, & Irons, 1995; Foxman & Beishuizen, 1999; Fuson *et al.*, 1997; Ruthven, 1998; Thompson & Smith, 1999). Several studies described two main categories of strategies – sequence-based and collections-based (e.g. Beishuizen & Anghileri, 1998; Cobb *et al.*, 1997; Foxman & Beishuizen, 2002; Thompson & Smith, 1999).

The standard example of a sequence-based strategy is the 'jump' strategy. Jump involves keeping the first number whole and adding (or subtracting) the second via a series of jumps. For example, a pupil might add 57 and 26 using jump by reasoning as follows: '57 and ten is 67, and ten more is 77; three more is 78, 79, 80; and three more make 83.' Researchers note that such sequence-based strategies depend on knowledge of sequential structures to jump by ten, and to make steps and hops in the number sequence (Fuson *et al.*, 1997; Treffers & Buys, 2001; Yackel, 2001). Classroom use of settings such as a number line that highlights the decades or a bead string with the decades demarked by colour (1–10 is blue, 11–20 is red, 21–30 is blue etc.) are linked to pupil use of sequential structure and sequence-based strategies (Klein, Beishuizen & Treffers, 1998).

The standard example of a collections-based strategy is the 'split' strategy. Split involves partitioning both numbers into tens and ones, adding (or subtracting) separately with the tens and the ones, and finally recombining the tens and ones subtotals. A pupil might add 57 and 26 using split by reasoning as follows: '50 and 20 are 70, 7 and 6 are 13, 70 and 13 make 83'. Collections-based strategies use collections-based structures (Fuson *et al.*, 1997; Treffers & Buys, 2001; Yackel, 2001). Classroom settings such as base-ten blocks are linked to the use of collections-based structures and strategies (Beishuizen, 1993).

Fuson *et al.* (1997) suggest that an advanced understanding of multi-digit addition and subtraction requires an integration of sequence-based and collections-based strategies. For example, an advanced pupil asked to add 5 doughnuts to 58 doughnuts might use a sequence-based strategy, jumping through 60 to 63, which is more efficient than split in this case; but when then asked how many boxes of ten she could fill, use her knowledge of collections-based structure to recognize 6 tens in 63.

Infrequency of sequence-based strategies among low-attaining pupils
Researchers have found that low-attaining pupils tend to use split strategies, indicating the development of knowledge of collections-based structure (Beishuizen, 1993; Foxman & Beishuizen, 2002). Research also suggests that many low-attaining pupils do not develop the strategy of jumping by tens and thus may not develop sequence-based structures (Beishuizen, 1993; Beishuizen *et al.*, 1997; Menne, 2001). Thus it is unlikely that these pupils can advance to integrated sequence-collections-based strategies which, we would argue, is important for number sense and mental computation.

Advantages of sequence-based strategies
Jump strategies can develop as abbreviations of pupils' informal counting strategies (Beishuizen & Anghileri, 1998; Olive, 2001). Following the view that pupils' knowledge should build on their informal strategies (Anghileri, 2001; Resnick, 1989), some researchers recommend teaching jump strategies (Klein *et al.*, 1998). A common difficulty with multi-digit addition and subtraction arises for pupils when they separate the digits in the tens place from the digits in the ones place and do not adequately regroup. For example, 57 + 26 is found to be '73' or even '713'. These difficulties arise in the case of split strategies but do not arise in the case of jump strategies (Beishuizen & Anghileri, 1998; Cobb, 1991; Fuson *et al.*, 1997). Beishuizen and colleagues found that pupils made significantly more errors when using split strategies than when using jump strategies. Importantly, even within a group of pupils identified as low-attaining, jump strategies were much more successful (Klein *et al.*, 1998). These results were confirmed by Foxman (2002). Studies comparing the use of split and jump strategies found that split led to more difficulty developing independence from concrete materials (Beishuizen, 1993), more procedural and conceptual confusion (Klein *et al.*, 1998) and slower response times, suggesting a heavier load on working memory (Wolters, Beishuizen, Broers & Knoppert, 1990). Subtraction tasks are a source of particular difficulties in multidigit arithmetic, and the potential confusions of subtraction using a split strategy are well documented. Confused responses using split suggest the collections-based structure offers a problematic representation of subtraction tasks (Fuson *et al.*, 1997). Success with split requires strong number sense and subtle insight into the procedure itself, whereas success with jump mainly requires knowing how to jump ten from any number (Beishuizen, 1993).

Developing flexibility with strategies
An important goal in improving multidigit number sense is flexibility with strategies, including recognising efficient short-cuts and making adaptations for unfamiliar problems (McIntosh *et al.*, 1992). Studies indicate that pupils more readily adapt the jump strategy to make efficient computation choices. According to Beishuizen *et al.*, this is due to 'the underlying mental representation of the number row up to 100' (1997). That is, using the sequential structure of number readily supports strategic insight into computation tasks.

In summary, pupils with arithmetic difficulties tend not to develop sequential structure and sequence-based strategies such as jump. It is likely that this denies them an integrated approach to multi-digit addition and subtraction, and access to the preferred strategies of arithmetically successful pupils. Further, development of sequential structure and strategies might resolve a number of typical multi-digit difficulties prevalent with collections-based strategies.

Assessment task groups and responses

As part of the project (referred to earlier in this article), 204 low-attaining pupils were interviewed twice during the school year to assess their number knowledge. The pupils were in third and fourth grades (8- to 10-year-olds) from a broad demographic range across the state of Victoria, and were selected for the study based on low results in screening tests. A method involving one to one, dynamic interview was used, in which the pupil is posed number tasks, and the interviewer pays close attention to the pupil's thinking process (Wright, Martland & Stafford, 2006). Interview assessments were recorded on videotape for later analysis.

We use the term 'task group' to refer to a group of closely related tasks used to investigate pupils' knowledge of a specific topic. In this paper, we discuss four task groups we found particularly valuable in assessing pupil knowledge of sequential structure:

1. Number word sequences by ones.
2. Number word sequences by tens.
3. Incrementing and decrementing by ten.
4. Locating numbers.

For each, we describe the range of low-attaining pupils' responses and difficulties, evident from analysis of the videotaped interviews.

Task group 1: Number word sequences by ones

Focus

Short sequences of number words, backwards and forwards; number word before or after; and bridging decades and hundreds.

Examples

'Count from 97. I'll tell you when to stop.' Stop at 113.
'Count backwards from 103.' Stop at 95.
'Say the number that comes just after 109'.
'Say the number that comes just before 100'.
Similarly for bridging 40, 210, 300, 990, 1000, 1100 forwards and backwards.

Low-attaining pupils' difficulties

Table 1 sets out examples of pupils' errors with number word sequences. Errors bridging 50 backwards indicate that the pupils have not fully constructed the number word sequence. Rather, they are aware of separated chains such as 41–49 and 51–59, and link these chains incorrectly when going backwards (Skwarchuk & Anglin, 2002). In the range 100 to 1000, number word sequence errors were common (Table 1). In many of the cases where pupils responded correctly to these tasks, their responses indicated a lack of certitude, particularly when bridging decade or hundred numbers. All of our low-attaining pupils made errors with number word sequences bridging 1000. Younger children's difficulties in establishing the number word sequence are well documented (e.g. Fuson, Richards & Briars, 1982; Wright, 1994). We have found the persistent errors and uncertainties of these older children striking. Our conclusion is that the assessment tasks described above are indicative of areas of knowledge that should be explicitly taught, at least in the case of low-attaining pupils.

Task group 2: Number word sequences by tens

Focus

Number word sequences by tens, forwards and backwards, on and off the decade.

Examples

'Count by tens.' Stop at 120.
'Count by tens from 24. I'll tell you when to stop.' Stop at 104.

Bridging 50 or 40 backwards
'52, 51, **40**, 49, 48…'
'52, 51, ∧ 49, 48…'
'52, 51, 50, **89**, 88…'
'42, 41, 40, **49**, 48…'
Bridging 100
'98, 99, 100, **ten hundred**'
'102, 101, ∧ 99, 98…'
Bridging 110 forwards
'108, 109, **1000**, 1001…'
'108, 109, **200**, 201, 202…'
Number word after 109: '**1000**'
Bridging 200
'198, 199. That's all I know.'
'198, 199, **1000**, 1001…'
'198, 199, ∧ 201, 202…'
'202, 201, ∧ 199, 198…'

Table 1: Pupil's errors in oral number word sequences. *Note*: Errors are marked in **bold**, omissions are marked with '∧'.

'Count by tens back from 52.' Stop at 2.
'Count by tens from 167.' Stop at 237.

Low-attaining pupils' responses and difficulties
The patterns of number word sequences by tens are inherent in the sequential structure of the base-ten number system. Jump strategies are derived from these patterns. Researchers have suggested that pupils can have difficulty producing number word sequences by tens off the decade, and hence be unable to develop a jump strategy (e.g. Beishuizen, 1993).

Skip counting by tens on the decade. All pupils interviewed could produce the sequence of decade numbers '10, 20, 30, …', although some had difficulty in continuing beyond 90.

Cannot count by tens from 24. Some pupils could not count by tens from 24. Responses included: (a) '24, 25, 20' and again '24, 25, 20?'; and (b) '24, 30, 34, 40'. One pupil could not count by tens from 24, but could count by tens from 25 – '25, 35, 45…'. It seems that these pupils' inability to make sense of the task arises from an unfamiliarity with sequences of tens off the decade compared with sequences of fives and of tens on the decade. Indeed, some of these pupils could count by tens on the decade up to 1000.

Counting by ones. When asked to count by tens from 24, some pupils counted each ten by ones. This could be laborious and sometimes unsuccessful. Sometimes, a pupil would seemed to become aware of the pattern they were producing, and their sequence would become more fluent perhaps curtailing the counting by ones.

Difficulties with teen numbers. Many pupils could not coordinate the teen numbers with a larger number sequence. When counting by tens back from 52 (52, 42, 32,…), some pupils had difficulty after 22. Responses included: (a) '… 22, 2'; (b) '…22, 14, 4'; and (c) '… 22, 10, 1'. Some were successful but their response involved counting back by ones after 22. Many pupils had difficulties with teen numbers in the hundreds, for example saying '336, 326, 316, **314**, 306, **304**.' A few pupils had difficulties with teens when counting back by tens on the decade: (a) '70, 60,…30, 20, **15**, 10.'; (b) '70, 60, …30, **12**, 10.'; and (c) '70, 60, 50, 40, 12, no, 20, 0? or 10?'. Irregularities in the names of teens mask their ten-structure (Fuson *et al.*, 1997) and this results in significant difficulties in saying sequences by tens.

Difficulties in the range 100 to 1000. Pupils who could skip count by ten off the decade in the range 1 to 100 experienced difficulties with bridging one hundred or higher hundred numbers. One pupil said '177, 187, 197, **one hundred and-**' then '**297**,' then '207, 217…'. When skip counting back by ten on the decade some pupils produced a sequence such as '430, 420, 410, **300**, 390, 380…'. This is analogous to a common error among younger pupils at decade numbers when counting backward by ones, for example, when counting back from 45, the pupil says, '45, 44, 43, 42, 41, **30**, 39, 38 …'. Pupils who erred when skip counting by saying 300 as the number ten less than 410, did not make a corresponding error when skip counting back by ten off the decade but there were difficulties at hundred numbers such as '336, 326, 316, 297' corrected to '296', and thus omitting 306. Another difficulty was discriminating the new hundreds number from the tens number, for example, an attempt to skip count by ten from 167 was '267, 367, 467'. When counting back by tens from 336, one response was: '326, 316, **312**' (pause), '326, **226**, 206, **two hundred and zero**, 196, 186, 176'. This sequence illustrates a persistent difficulty with the teen numbers within the hundreds, and confusion when bridging 200. It is clear that knowledge of sequences of tens beyond 100 is a significant extension of knowledge of sequences of tens up to 100.

The responses described above are indicative of weaknesses in pupils' knowledge of the sequential structure of numbers.

We believe it is important to address these weaknesses through intervention.

Task group 3: Incrementing and decrementing by 10

Focus

Ten more and ten less, on and off the decade, and bridging decades and hundreds.

Examples

Show '20' on a card. Ask 'Which number is ten more than this?'

Similarly for 79, 356, 195, 999.

Show '30' on a card. Ask 'Which number is ten less than this?'

Similarly for 79, 356, 306, 1005.

Relationship to knowledge of the tens structure of the number sequence

We were interested in whether pupils could solve the tasks in Task Group 3 without counting by ones or trying to use an algorithm for addition or subtraction. We would regard such pupils as having knowledge of the tens structure of the number sequence.

The tasks of (a) incrementing and decrementing by ten and (b) skip counting by ten seemed to be linked in the sense that pupils showed similar levels of advancement in their responses to these task groups. Nevertheless, the tasks involving incrementing or decrementing by ten are distinct from tasks of skip counting by ten. There was incongruence in pupils' responses on these two task groups. For example, one pupil skip counted by tens from 167 successfully: '177, 187, 197, 297' self-corrected to '207, 217…', but could not solve ten more than 195: 'one hundred and…' changed to '220?' changed to '225'.

Pupils might construe the incrementing task as an addition task rather than a task based on a number sequence with increments of ten. Thus they are unable to regard ten more as one increment of a sequence with increments of ten. Even if they can regard ten more as one increment in a sequence with increments of ten, they are apparently unable to increment the sequence from a standing start, when the increment involves bridging a hundred number. Alternatively, we could say that a pupil who can increment by ten to bridge a hundred number has constructed a sequence-based strategy for the operation of adding ten.

Progressions in incrementing and decrementing by ten

Pupils' success with tasks involving incrementing or decrementing by ten tended to progress as follows:

- 2-digit off-the-decade: 'ten more/less than 79',
- 3-digit off-the-decade: 'ten more/less than 356',
- forward across a hundred number: 'ten more than 195',
- backward across a hundred number: 'ten less than 306',
- forward across 1000: 'ten more than 999',
- backward across 1000: 'ten less than 1005'.

Thus a pupil who was successful at the third progression (starting from the uppermost progression) was likely to succeed with the tasks at the first two progressions and not succeed with the tasks from the fourth progression onward. Some pupils could not increment by ten off the decade at all. Difficulties with the teen number sequence were also evident in these tasks, for example, a pupil could find ten more than 356, but not find ten more than 306. Tasks involving 1000, that is ten more than 999 and ten less than 1005, were especially difficult for virtually all of the pupils.

'Which number is ten less than 306?'

The task of finding ten less than 306 was particularly difficult for many pupils. Pupils' responses included:

- 'I don't know.'
- Incorrect counting: '210?', '299', '300'.
- Counting back by ones with an incorrect sequence, and using fingers to keep track of the ten counts: '305, 304, 303, 302, 301, 330, 329, 328, 327, 326!'.
- Counting back by ones and answering '295'.
- Counting back by ones successfully.
- Jumping back ten successfully.

To solve this task by jumping back ten requires knowing the decade before 301 is the 290s. Many of these pupils did not know this or could not apply this knowledge to solve the task, that is, they lacked knowledge of the sequential structure of numbers. Further, many of the pupils did not count back by ones to solve this task. Apparently these pupils could not construct a representation of this problem that was embedded in the number sequence. Those pupils who attempted to count back on this task were consistently more successful on other tasks involving incrementing by ten than pupils who did not attempt to count back.

Task group 4: Locating numbers on a number line

Focus

Locating numbers on a linear representation of the number sequence from 0 to 100.

Example

Pupil is given a pen, and a line on paper with only the endpoints 0 and 100 labeled (see Figure 1). Pupil is asked to

'Mark where 50 is on the line,' and then 'Label that as 50.' Similarly mark and label 25; 98; 62.

Low-attaining pupils' responses and difficulties

A locating number task requires knowledge of the number sequence. To locate the numbers efficiently requires using ideas such as:

- 50 at half-way;
- 25 at half-way to 50;
- 98 at two steps before 100; and
- 62 just after 60, which is ten after 50.

To do the task well also requires knowledge of linear measure and proportion which may be somewhat distinct from number sequence knowledge. Pupils' difficulties with this task indicate a lack of knowledge of sequential structure and also a lack of knowledge of linear measure. We have found pupils' responses to be interesting, and revealing of their number sequence knowledge. Four examples are discussed below.

Renee's response, shown in Figure 2, is typical. It would seem there is some sense of global location: 50 is placed at half-way; 25 placed perhaps from a sense of decades, or from half of 50; 98 is probably located to be near 100, but with a weak sense of the measure of the 2-step gap.

A weaker response can be very revealing. Helen (see Figure 2) does find 50 as 'half way'. But to locate 25, she marks all the ones from 0 to 25. She does not count in tens, though she does emphasise her '20' point. Her 25 ends up almost at 50, and it is not clear whether she regards this as problematic. She locates 98 two steps back from 100, but the steps are too big. Helen is using an aspect of the number structure, but is not checking against another aspect, that is, 98 as '8 more than 90'. She does not seem to have a global or embedded sense of the structures of the sequence. To locate 62, Helen again counts by ones. She does count on from 50, and she emphasises the '60' point along the way, showing some appreciation of how the number structure can support her solution. But she does not curtail counting by ones.

Nate finds each of his numbers by counting and marking fives (see Figure 2). He locates 98 after counting to 95, and locates 62 just past 60. He does not count by ones, but doesn't regard the decades as reference points. Rather, he counts by fives. Perhaps more striking is that, in contrast to Helen, Nate finds both 62 and 98 by counting by fives from five. He marks 98 just short of 100 but does not use 100 as the reference to locate 98. He does seem concerned to line up the successive counts to 50 at the same 50 mark, but he does not simply count-on from 50—he begins afresh from five each time. His approach is analogous to counting-all rather than counting-on to solve an addition task. Both Helen and Nate frequently counted by ones on their fingers to solve addition

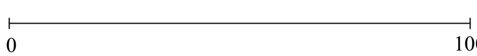

Figure 1: Blank number line for 'locating numbers' task

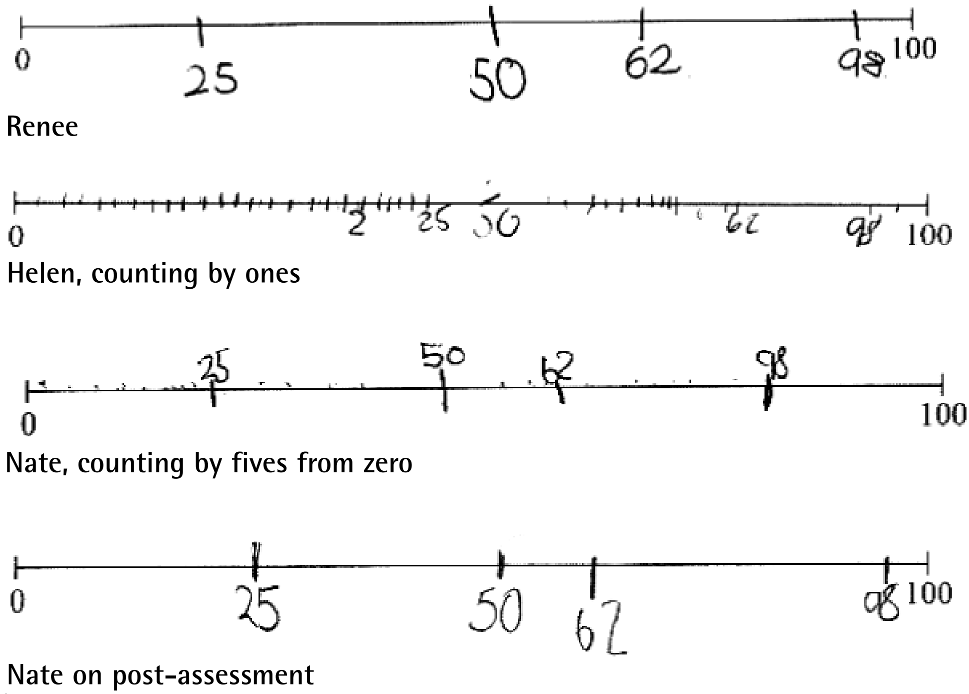

Figure 2: Locating numbers on a number line

tasks, and neither could skip count by tens off the decade. Their responses on this Task Group indicate a lack of knowledge of structures in the number sequence.

Both of these pupils received intensive individual intervention instruction after these assessments. The instruction did not focus on tasks of locating numbers but included a significant focus on counting by ones and tens, and on recording on an empty number line, additions using a jump strategy. When assessed with the locating number task after the intervention period, Helen was no longer marking ones, and she located 25 appropriately. Nate's post-assessment response is shown in Figure 2. His pen moves in jumps of ten and one. This learning that Nate demonstrated in his post-assessment can be attributed to the instruction that focused on recording jump strategy additions on a number line. He shows a clear use of decade structure, and he is no longer working from one. Interestingly, his knowledge of linear measure and proportion has also advanced. Thus, without explicit instruction on locating numbers, his broad development of number sequence knowledge has made significant differences to his responses on the task group of locating numbers.

This Task Group is very useful because it can reveal knowledge of number sequence structure, can differentiate levels of understanding, and can enable learning over time to be documented.

Conclusion

We claim that the sequential structure of numbers is important basic number knowledge. We advocate that pupils' number learning should include a focus on number word sequences up to 1000, skip counting and incrementing by tens off the decade, and locating numbers in the range 1 to 100. It is striking that many third and fourth grade pupils (aged 8 to 10 years) are not successful on the assessment tasks described in this report. In our view, a focus on sequential structure exemplifies an informed approach to tackling numeracy difficulties (Dowker, 2005).

Studies suggest that weakness in these sequence-based tasks is characteristic of low-attaining pupils (Beishuizen *et al.*, 1997; Menne, 2001). Our study accords with this. We recommend that low-attaining pupils be assessed for knowledge of sequential structure, and that intervention include explicit attention to development of this knowledge. The four assessment task groups discussed in this report can inform detailed assessment of pupils' number sequence knowledge. We are developing instructional activities for this topic in our current research project with low-attaining pupils, trialling, for example, flexible incrementing and decrementing by tens and ones (Wright, Martland, Stafford & Stanger, 2002) and jumping on an empty number line (Menne, 2001). We are also developing activities targeting the pupils' development of the related sequence-based mental strategies for addition and subtraction.

There has been considerable discussion of pupil and curriculum choices between collections-based and sequence-based strategies for addition and subtraction (Beishuizen, 2001). Studies suggest that low-attaining pupils can have more success with sequence-based addition strategies, such as jump, than with collection-based strategies, such as split (Beishuizen, 1993). Importantly, if teachers choose to emphasise jump, pupils will require a co-development of knowledge of sequential structure (Menne, 2001). Further, regardless of choice of arithmetic strategy (jump or split) our curriculum should recognise the importance of sequential structure as a basic aspect of number.

Acknowledgements

The authors gratefully acknowledge the support for this project from the Australian Research Council under grant LP0348932 and from the Catholic Education Commission of Victoria. As well, the authors acknowledge the contributions of Gerard Lewis and Cath Pearn (partner investigators) to this project. Finally, the authors express their sincere thanks to the teachers, pupils and schools participating in the project.

Address for correspondence

School of Education, Southern Cross University, PO Box 157, Lismore NSW 2480, Australia
E-mail: Bob.Wright@scu.edu.au

References

Anghileri, J. (2000). *Teaching number sense.* London: Continuum.

Anghileri, J. (2001). Intuitive approaches, mental strategies and standard algorithms. In J. Anghileri (Ed.), *Principles and practices in arithmetic teaching – Innovative approaches for the primary classroom.* Buckingham: Open University Press.

Askew, M., Brown, M., Rhodes, V., Wiliam, D. & Johnson, D. (1997). *Effective teachers of numeracy: Report of a study carried out for the Teacher Training Agency.* London: King's College, University of London.

Beishuizen, M. (1993). Mental strategies and materials or models for addition and subtraction up to 100 in Dutch Second Grades. *Journal for Research in Mathematics Education, 24*(4), 294–323.

Beishuizen, M. (2001). Different approaches to mastering mental calculation strategies. In J. Anghileri (Ed.), *Principles and practices in arithmetic teaching* (pp. 119–130). Buckingham: Open University Press.

Beishuizen, M. & Anghileri, J. (1998). Which mental strategies in the early number curriculum? A comparison of British ideas and Dutch views. *British Educational Research Journal, 24*(3), 519–538.

Beishuizen, M., Van Putten, C. M. & Van Mulken, F. (1997). Mental arithmetic and strategy use with indirect number problems up to one hundred. *Learning and Instruction, 7*(1), 87–106.

Cobb, P. (1991). Reconstructing elementary school mathematics. *Focus on Learning Problems in Mathematics, 13*(2), 3–22.

Cobb, P., Gravemeijer, K.P.E., Yackel, E., McClain, K. & Whitenack, J. (1997). Mathematizing and symbolizing: the emergence of chains of signification in one first-grade classroom. In D. Kirshner & J.A. Whitson (Eds.), *Situated Cognition Theory:*

Social, semiotic and neurobiological perspectives (pp. 151–233). Mahwah, NJ: Lawrence Erlbaum.

Cooper, T.J., Heirdsfield, A. & Irons, C.J. (1995). Years 2 and 3 children's strategies for mental addition and subtraction. In B. Atweh & S. Flavel (Eds.), *Proceedings of the eighteenth annual conference of the Mathematics Education Research Group of Australasia* (pp. 195–202). Darwin: Mathematics Education Research Group of Australasia.

Dowker, A. (2005). *Individual differences in arithmetic: Implications for psychology, neuroscience and education.* Hove, UK: Psychology Press.

Foxman, D. & Beishuizen, M. (1999). Untaught mental calculation methods used by 11-year-olds. *Mathematics in School*, 5–7.

Foxman, D. & Beishuizen, M. (2002). Mental calculation methods used by 11-year-olds in different attainment bands: A reanalysis of data from the 1987 APU Survey in the UK. *Educational Studies in Mathematics, 51*(1–2), 41–69.

Fuson, K.C., Richards, J. & Briars, D. (1982). The acquisition and elaboration of the number word sequence. In C.J. Brainerd (Ed.), *Progress in Cognitive development: Vol. 1 Children's logical and mathematical cognition* (pp. 33–92). New York: Springer-Verlag.

Fuson, K.C., Wearne, D., Hiebert, J., Murray, H., Human, P., Olivier, A., Carpenter, T.P. & Fennema, E. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education, 28*, 130–162.

Hiebert, J. & Wearne, D. (1996). Instruction, understanding, and skill in multidigit addition and subtraction. *Cognition and Instruction, 14*(3), 251–283.

Klein, A.S., Beishuizen, M. & Treffers, A. (1998). The empty number line in Dutch second grades: Realistic versus gradual program design. *Journal for Research in Mathematics Education, 29*(4), 443–464.

McIntosh, A., Reys, B.J. & Reys, R.E. (1992). A proposed framework for examining basic number sense. *For the Learning of Mathematics, 12*, 2–44.

Menne, J. (2001). Jumping ahead: an innovative teaching program. In J. Anghileri (Ed.), *Principles and practices in arithmetic teaching – Innovative approaches for the primary classroom* (pp. 95–106). Buckingham: Open University Press.

Olive, J. (2001). Children's number sequences: An explanation of Steffe's constructs and an extrapolation to rational number of arithmetic. *The Mathematics Educator, 11*(1), 4–9.

Resnick, L.B. (1989). Developing mathematical knowledge. *American Psychologist, 44*, 162–169.

Ruthven, K. (1998). The use of mental, written, and calculator strategies of numerical computation by upper primary students within a 'calculator aware' number curriculum. *British Educational Research Journal, 24*(1), 21–42.

Skwarchuk, S. & Anglin, J.M. (2002). Children's aquisition of the English cardinal number words: A special case of vocabulary development. *Journal of Educational Psychology, 94*(1), 107–125.

Sowder, J.T. (1992). Making sense of numbers. In G. Leinhardt, R. Putnam, & R.A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 1–51). Hillsdale, NJ: Lawrence Erlbaum.

Thompson, I. (Ed.) (1997). *Teaching and learning early number.* Buckingham: Open University Press.

Thompson, I. & Smith, F. (1999). *Mental calculation strategies for the addition and subtraction of 2-digit numbers (Report for the Nuffield Foundation).* Newcastle upon Tyne: University of Newcastle upon Tyne.

Treffers, A. (1991). Didactical background of a mathematics program for primary education. In L. Streefland (Ed.), *Realistic mathematics education in primary school* (pp. 21–56). Utrecht: Freudenthal Institute.

Treffers, A. & Buys, K. (2001). Grade 2 and 3– Calculation up to 100. In M. van den Heuvel-Panhuizen (Ed.), *Children learn mathematics.* The Netherlands: Freudenthal Institute, Utrecht University.

Wolters, G., Beishuizen, M., Broers, G. & Knoppert, W. (1990). Mental arithmetic: Effects of calculation procedure and problem difficulty on solution latency. *Journal of Experimental Child Psychology, 49*, 20–30.

Wright, R.J. (1994). A study of the numerical development of 5-year-olds and 6-year-olds. *Educational Studies in Mathematics, 26*, 25–44.

Wright, R.J., Martland, J. & Stafford, A.K. (2006). *Early numeracy: Assessment for teaching and intervention* (2nd ed.). London: Paul Chapman Publishing.

Wright, R.J., Martland, J. Stafford, A.K. & Stanger, G. (2002). *Teaching number: Advancing children's skills and strategies.* London: Paul Chapman Publishing.

Yackel, E. (2001). Perspectives on arithmetic from classroom-based research in the United States of America. In J. Anghileri (Ed.), *Principles and practices in arithmetic teaching – Innovative approaches for the primary classroom* (pp. 15–31). Buckingham: Open University Press.